Math 10B - Calculus of Several Variables II
Quiz 2
April 22, 2011

Name: _____

Total
/10

Evaluate the integral:

$$\int_0^{\pi}\int_0^{2\pi}\int_1^3 \rho^2 \sin\varphi \; d\rho d\theta d\varphi.$$

Extra credit: Thinking of this as an integral in spherical coordinates, what is the region it gives the volume of? (The extra credit will not raise your quiz score past 10/10.)

Solution.

$$\begin{aligned}
\int_0^{\pi}\int_0^{2\pi}\int_1^3 \rho^2 \sin\varphi \; d\rho d\theta d\varphi &= \int_0^{\pi}\int_0^{2\pi} \left(\frac{1}{3}\rho^3\right)\Big|_1^3 \sin\varphi \; d\theta d\varphi \\
&= \int_0^{\pi}\int_0^{2\pi} \left(9 - \frac{1}{3}\right)\sin\varphi \; d\theta d\varphi \\
&= \frac{26}{3}\int_0^{\pi}\int_0^{2\pi} \sin\varphi \; d\theta d\varphi \\
&= \frac{26}{3}\int_0^{\pi} \theta\Big|_0^{2\pi} \sin\varphi \; d\varphi \\
&= \frac{52\pi}{3}\int_0^{\pi} \sin\varphi \; d\varphi \\
&= \frac{52\pi}{3}(-\cos\varphi)\Big|_0^{\pi} \\
&= \frac{52\pi}{3}(-\cos\pi + \cos 0) \\
&= \frac{104\pi}{3}
\end{aligned}$$

Recall that, when changing to spherical coordinates, we have $dV = \rho^2 \sin\varphi \; d\rho d\theta d\varphi$. Since the bounds on the integral define the region between the sphere of radius 1 and the sphere of radius 3, the integral gives the volume of this "spherical shell" (or 3 dimensional annulus if you like). Described explicitly, the region would be:

$$\{(x, y, z) \in \mathbb{R}^3 \mid 1 \leq x^2 + y^2 + z^2 \leq 9\}.$$

□